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DISCUSSION OF RATE OF CHANGE OF GRADE PER STATION

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By T. F. Hickerson, Robert T. Howe, E. N. Prouty, and Clarence J. Brownell

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Technical Division	*Proceedings-Separate Number*
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City Planning	58, 60, 62, 64, 93, 94, 99, 101, 104, 105, 115, 131, 138, 148, 151, 152, 154, 164, 167, 171, 172, 174 (Discussion: D-65, D-86, D-93, D-99, D-101, D-105, D-108, D-115, D-117)
Construction	154, 155, 159, 160, 161, 162, 164, 165, 166, 167, 168 (Discussion: D-75, D-92, D-101, D-102, D-109, D-113, D-115, D-121)
Engineering Mechanics	142, 143, 144, 145, 157, 158, 160, 161, 162, 169 (Discussion: D-24, D-33, D-34, D-49, D-54, D-61, D-96, D-100, D-122, D-125, D-127)
Highway	138, 144, 147, 148, 150, 152, 155, 163, 164, 166, 168 (Discussion: D-103, D-105, D-108, D-109, D-113, D-115, D-117- D-121)
Hydraulics	141, 143, 146, 153, 154, 159, 164, 169, 175 (Discussion: D-90, D-91, D-92, D-96, D-102, D-113, D-115, D-122)
Irrigation and Drainage	129, 130, 133, 134, 135, 138, 139, 140, 141, 142, 143, 146, 148, 153, 154, 156, 159, 160, 161, 162, 164, 169, 175 (Discussion: D-97, D-98, D-99, D-102, D-109, D-117)
Power	120, 129, 130, 133, 134, 135, 139, 141, 142, 143, 146, 148, 153, 154, 159, 160, 161, 162, 164, 169, 175 (Discussion: D-96, D-102, D-109, D-112, D-117)
Sanitary Engineering	55, 56, 87, 91, 96, 106, 111, 118, 130, 133, 134, 135, 139, 141, 149, 153, 166, 167, 175 (Discussion: D-96, D-97, D-99, D-102, D-112, D-117)
Soil Mechanics and Foundations	43, 44, 48, 94, 102, 103, 106, 108, 109, 115, 130, 152, 155, 157, 166 (Discussion: D-86, D-103, D-108, D-109, D-115)
Structural	133, 136, 137, 142, 144, 145, 146, 147, 150, 155, 157, 158, 160, 161, 162, 163, 164, 165, 166, 168, 170, 175 (Discussion: D-51, D-53, D-54, D-59, D-61, D-66, D-72, D-77, D-100, D-101, D-103, D-109, D-121, D-125, D-127)
Surveying and Mapping	50, 52, 55, 60, 63, 65, 68, 121, 138, 151, 152, 172, 173 (Discussion: D-60, D-65)
Waterways	120, 123, 130, 135, 148, 154, 159, 165, 166, 167, 169 (Discussion: D-8, D-9, D-19, D-27, D-28, D-56, D-70, D-71, D-78, D-79, D-80, D-112, D-113, D-115)

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DISCUSSION

T. F. HICKERSON,[11] M. ASCE.—In urging a more general adoption of "rate of change of grade" for highways, as has long been the practice for railways, this paper is timely. A clearer concept of the sharpness of the curve is obtained and, incidentally, there is provided greater facility in the solution of curve problems.

The determination of the vertical alinement for a pair of reverse curves (Example 5) is a noteworthy contribution; but it appears simpler to substitute the given data in Eqs. 9 to 12 and solve them simultaneously, rather than to use the long expressions presented in cases II to VII.

Using the author's sign convention, it follows that the rate of change of grade is positive for sags and negative for crests; hence, $R_2 - R_1$, shown in Fig. 5, is positive.

The derivation of Eq. 20 is unique. The absolute value of R must be used in this formula, otherwise it reads

$$S = \frac{6}{\sqrt{-R}} \qquad (40)$$

ROBERT T. HOWE,[12] A. M. ASCE.—The theory of vertical curves is useful to a teacher of railroad and highway surveying, as a review of one phase of analytic geometry. The topic may be introduced through the general equation of a parabola with its axis parallel to the y-axis—

$$y = A\,x^2 + B\,x + C \qquad (41)$$

in which x is the horizontal distance from the origin (selected through the BVC) to any point on the curve, and y is the elevation of that point.

Comparing the first two coefficients in Eq. 41 with those of Eq. 4*a*, coefficient A is found to equal $R/2$ and B is found to equal G_1. Referring to Fig. 1, it is found that C is the elevation of BVC.

Use the nomenclature of Eq. 4*b*, and let h equal the distance from the zero datum (usually sea level) to the directrix of the parabola. The basic properties of the parabola give the equation—

$$y - h = \sqrt{(x - a)^2 + [y - (h + p)]^2} \qquad (42)$$

—which reduces to

$$y = \frac{x^2}{2\,p} - \frac{2\,a}{2\,p}\,x + \frac{a^2 + 2\,p\,h + p^2}{2\,p} \qquad (43)$$

NOTE.—This paper by Clarence T. Brownell was published in March, 1952, as *Proceedings-Separate No. 121*. The numbering of footnotes, equations, illustrations, and tables in this Separate is a continuation of the consecutive numbering used in the original paper.

[11] Prof. of Applied Math., Univ. of North Carolina, Chapel Hill, N. C.

[12] Asst. Prof. of Civ. Eng., Univ. of Cincinnati, Cincinnati, Ohio.

Consider Fig. 1(*a*) as an example, in which the vertex is outside the vertical curve, $G_1 = +1\%$, $G_2 = +7\%$, and $L = 6$ stations, from which $R = +1\%$. Assume that the BVC is at El. 200.00 rather than at El. 0. Then, $\frac{1}{2p} = \frac{R}{2} = \frac{1}{2}$, from which $p = +1.00$, and $-\frac{2a}{2p} = G_1 = 1.00$, from which $a = -1.00$. This means that the vertex is 1 station to the left of the origin, or at station 9 + 00. Then $\frac{a^2 + 2ph + p^2}{2p} = 200.00$, the elevation of the BVC; thus, $h = 199.00$ ft, the elevation of the directrix. Adding $\frac{p}{2}$ to h yields 199.50 ft, the elevation of the vertex of the parabola.

In Table 1, the author gives an illustration of a situation in which the vertex of the parabola is within the limits of the vertical curve. Here, $G_1 = -1.10\%$, $G_2 = +1.30\%$ and $L = 6$ stations from which $R = +0.4\%$. The BVC is at El. 100.00. Then

$$y = 0.2\,x^2 - 1.10\,x + 100.00 \quad\ldots\ldots\ldots\ldots\ldots\ldots (44)$$

which yields $p = +2.50$, $a = 2.75$, and $h = 97.237$ ft. The vertex and low point of the vertical curve are therefore at station 12 + 75.00, at El. 98.487.

TABLE 7.—COMPUTATION OF INTERMEDIATE ELEVATIONS, BASED ON EQ. 44

Station (1)	Description (2)	x, from BVC (3)	$-1.10\,x$ (4)	x^2 (5)	$0.2\,x^2$ (6)	Elevation from BVC[a] (7)	Elevation,[b] E_2 (8)
12+90	Pier No. 3	2.90	−3.19	8.41	1.68	−1.51	98.49
12+20	Abutment No. 1	2.20	−2.42	4.84	0.97	−1.45	98.55
10+00	Beginning of vertical curve (BVC)	0	0	0	0	0	100.00

[a] The values in Col. 7 are found by adding the values in Col. 4 to those in Col. 6. [b] The values in Col. 8 are found by adding 100.00 ft to the values in Col. 7.

The author's solution for the elevations of intermediate points of this curve is novel, but certainly no more direct than the tabular solution of the equation of the curve, as demonstrated in Table 7.

Should the grade at any station be desired, differentiate the equation of the curve, as in Eq. 6*a*, and then substitute the proper value of x.

The solution of a general equation of the form of Eq. 41 also simplifies the selection of a curve that will meet certain restrictions, such as passing through a given elevation at a specified station. For example, if it is given that $G_1 = +2.30\%$, $G_2 = +4.70\%$, the BVC is at El. 218.75 and station 13 + 00; and there is a fixed point at station 17 + 40 of El. 230.13; let it be required to find the length of the vertical curve that will fit this situation. The general equation of this curve becomes

$$y = \frac{2.40}{2\,L}\,x^2 + 2.30\,x + 218.75 \quad\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

At the fixed point, $y = 230.13$ and $x = 4.40$ stations. Therefore, $L = 18.438$ stations.

Although the author's development of the rate-of-change-of-grade method is interesting, and undoubtedly has advantages in certain situations, it is perhaps a step away from the fundamentals that the colleges are so frequently urged to stress. Because this rate-of-change-of-grade method rests squarely on two of the three right-hand terms of the general equation of a parabola with its axis parallel to the y-axis, why not simply use the entire equation as illustrated herein.

E. N. Prouty,[13] A.M. ASCE.—The solution proposed in this paper can be simplified considerably. For example, in the equation for a vertical curve the elements given in any problem are G_1 and G_2, the rates of the intersecting grades to be joined. Those to be selected to satisfy the conditions of the particular location of construction and operation are L, the length, and R, the rate of change of grade per station. From these, the most rapid and simplest solution would seem to be to use Eq. 2 to get V for the chosen L; then, combining Eqs. 1 and 2 (see Eq. 8)—

$$V = L^2 \frac{R}{2} \quad \text{(46)}$$

—from which R for that particular length L is fixed. The rate of change in grade R is constant for any curve of length L. It may be used advisedly as the nominal characteristic of that curve and must satisfy the requirements for safety and comfort of traffic under the conditions of volume and speed provided for that curve.

The ordinate from the tangent having grade G to the curve at distance x from the BVC, being $\frac{V}{L^2} x^2$, allows a quick examination of curve point elevation at any particular x without computing all consecutive chord points from the BVC. When curve elevations at selected points prove to be satisfactory, then a full tabulation, $x_1, x_2, \ldots$, of station points on the curve, where grade elevations are required to be set for construction—elevations of the tangent at $G_1, x_2, x_2, \ldots$, values of $x^2{}_1, x^2{}_2, \ldots$, values of $\frac{V}{L^2} x^2{}_1, \frac{V}{L^2} x^2{}_1, \ldots$, and grade elevations on the curve at $x_1, x_2, \ldots x_L$—is a form of record more readily adaptable for field checking and use than is Table 1.

C. J. Brownell,[14] A. M. ASCE.—In reply to Mr. Howe's suggestion concerning the fundamental equation of the parabola, the one important characteristic of a parabola is that the change of slope per unit of distance is constant. It is the rate of this change of slope that distinguishes one parabola from another. In terms of analytic geometry, this rate is expressed by the factor $1/p$. In the vernacular of the highway engineer, it is the rate of change of

[13] Engr., Clyde C. Kennedy, San Francisco, Calif.

[14] Associate Bridge Engr., Bridge Dept., State Div. of Highways, Sacramento, Calif.

grade per station—designated R in this paper. The description and solution of vertical curves by the rate of change of grade per station are steps toward the practical application of mathematical theory.

With reference to Mr. Prouty's remarks relative to Eq. 45, R is the distinguishing characteristic of a vertical curve of infinite length, and L can be any segment of the curve as illustrated in Fig. 12.

Mr. Howe's use of sea-level datum as the x-axis is an excellent way of explaining the complete problem. The use of h, the vertical distance from sea level to the directrix of the parabola, would be good for impressing the theory of the parabola on students. However, in highway practice the directrix and focus of the parabolic vertical curve are almost never referred to nor considered. The elevation E_1 expresses the distance from sea level to the first point on the curve (the intersection of the curve with the y-axis).

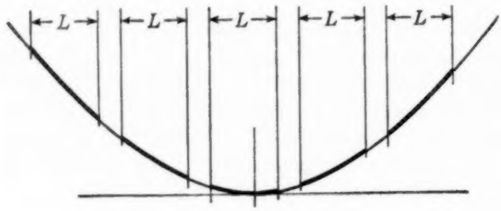

FIG. 12.

Mr. Hickerson's observation concerning the use of the absolute value of R in Eq. 20 is correct. However, a better expression would be

$$S = 6\sqrt{\frac{-1}{R}} \quad \text{.......................(47)}$$

in which R has its proper algebraic sign. Similarly, under the heading, "Nonpassing Sight Distance," the values of V are -4.5 ft and -0.33 ft. The subsequent expressions containing these values are changed accordingly. Eq. 28 should read

$$S = L_o + L_c = (0.8165 + 3)\sqrt{\frac{-1}{R}} = 3.8165\sqrt{\frac{-1}{R}}.$$

The signs in the ensuing expressions must be similarly changed, so that $S = \sqrt{\frac{-14.662}{R}}$ and $L_c = \sqrt{\frac{-3.667}{R}}$, and the value of V_c becomes -1.833.

For summit vertical curves, the value of V is always negative, as it is measured downward from the tangent. Figs. 7 and 8, "Sight Distance Over a Summit," would have been more appropriately drawn in the fourth quadrant.

Mr. Howe's discussion indicates the desirability of tabulating the solution in terms of the equation. In keeping with this procedure, the tabular system used in Tables 1 and 2 can be expressed as

$$\frac{G_1 + (G_1 + L\,R)}{2}\,L + E_1 = E_2 \text{ (Col. 8)}$$

in which

$$L = \text{Distance between points (Col. 3)}$$

$$L\,R = G_2 - G_1 \text{ (Col. 4)}$$

$$G_1 + L\,R = G_2 \text{ (Col. 5)}$$

$$\frac{G_1 + (G_1 + L\,R)}{2} = \frac{G_2 + G_1}{2} \text{ (Col. 6)}$$

$$\frac{G_1 + (G_1 + L\,R)}{2} L = H \text{ (Col. 7.)}$$

In reply to Mr. Prouty's suggestion concerning comfort and safety, reference is made to a solution of this problem by Professor Ralph A. Moyer.[15] Converting his equation for comfort on either sag or summit vertical curves to the nomenclature of this paper $R_{\text{comfort}} = \frac{15{,}000}{(\text{Velocity})^2}$. His expression for take-off speed reduces to $R_{\text{take-off}} = \frac{150{,}000}{(\text{Velocity})^2}$. In these expressions the velocity is in miles per hour.

TABLE 8.—COMPARISON OF COMPUTATION TIMES

Computer	Time, in Minutes: Tangent-offset method	Time, in Minutes: Rate-of-change-of-grade-per station method	Time saving, in %
	Problem 1A		
R. W. B.	19	14	26
R. P. H.	20	10	50
W. J. J.	18	17	5
C. J. B.	16	14	12
Average	18.25	13.75	24.5
	Problem 1B		
R. W. B.	18	12	33
R. P. H.	20	16	20
W. J. J.	10	10	0
C. J. B.	19	13	31
Average	16.75	12.75	23.8
Over-all average	17.5	13.25	24.3

In connection with Mr. Prouty's remarks, a brief time study was conducted among a few engineers who are familiar with both the methods of tangent offset and rate of change of grade per station for the solution of vertical curves. Two problems of the type of Case 1, having six intermediate points each, but having different stations, grades, and R-values, were used. The results of this study are tabulated in Table 8. The averages show that 24.3% less time was re-

[15] *Supplementary Notes and Typical Problems for Highway Engineering Course, Civil Engineering 106, University of California Syllabus Series, Syllabus W C*, by Ralph A. Moyer and John Hugh Jones, Univ. of California Press, Berkeley, Calif., (2d Ed. Revised) 1950, p. 120.

quired by the method of rate of change of grade per station than by the method of tangent offsets for a single solution of the problem. One solution by the method of rate of change of grade per station is sufficiently accurate because the computations close on the EVC. However, a solution by tangent offsets requires a second independent calculation for a check. If the average time for a single solution by tangent offsets is doubled to account for this check, there results a net saving of 62.1% in favor of the method of rate of change of grade per station.

Corrections for Transactions.—The sentence containing Eq. 4*b* should read "* * * would be written

$$y = \frac{-a_x}{p} + \frac{1}{2\,p}\,x^2 \qquad \text{(4b)}$$

in which * * * p is the distance from the directrix to the focus * * *." The following sentence should read "* * * and $G_1 = \frac{-a}{p}$, from which

$$a = \frac{-G_1}{R} \qquad \text{(5)."}$$

Fig. 6(*b*) should read "$S > L$, Passing Sight Distance"; p. 13, line 1, read "* * * when applied to Figs. 7 and 8 * * *."